On certain sums over the nontrivial zeta zeros

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Abstract

We study coefficients b_n that are expressible as sums over the Li/Keiper constants λ_j. We present a number of relations for and representations of b_n. These include the expression of b_n as a sum over nontrivial zeros of the Riemann zeta function, as well as integral representations. Conditional on the Riemann hypothesis, we provide the asymptotic form of $b_n \sim 2^{-n-2} \ln n$.

Key words and phrases

Riemann zeta function, Li criterion, Li/Keiper constants, Riemann hypothesis, functional equation, nontrivial zeros, integral representation, asymptotic form

MSC numbers

11M06, 11M26, 11Y60

Introduction

Let ζ denote the Riemann zeta function, and $\xi(s) = (s/2)(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ the classical completed zeta function, where Γ is the Gamma function [9, 12, 16, 17]. Within the critical strip $\mathcal{S}$, $0 < \text{Re } s < 1$, the complex zeros of ζ and ξ coincide, and we denote them by ρ. The ξ-function is entire, of order 1, and of maximal type.

Herein, we mainly investigate certain sums b_n over complex zeta function zeros. We provide various representations and properties of these sums. We also supply some remarks on the Li criterion [14] for the Riemann hypothesis (RH).

We recall that the Li equivalence for the RH results as a necessary and sufficient condition that the logarithmic derivative of the function $\xi[1/(1-z)]$ be analytic in the unit disk. This obtains from a conformal map of the critical strip to this disk. The equivalence [14] states that a necessary and sufficient condition for the nontrivial zeros of the ζ-function to lie on the critical line Re $s = 1/2$ is that constants $\{\lambda_k\}_{k=1}^{\infty}$ are nonnegative for every integer k. The sequence $\{\lambda_n\}_{n=1}^{\infty}$ can be defined by

$$\lambda_n = \frac{1}{(n-1)!}\frac{d^n}{ds^n}[s^{n-1}\ln \xi(s)]_{s=1}. \tag{1.1}$$

The λ_j's are connected to sums over the nontrivial zeros of $\zeta(s)$ by way of [13, 14]

$$\lambda_n = \sum_\rho \left[1 - \left(1 - \frac{1}{\rho}\right)^n\right]. \tag{1.2}$$

For further discussion of the Li criterion, its application, and results on series expansion of the ξ function, see for instance [2, 4, 5, 6, 3, 7].

In particular, consider the expansions [11]

$$\ln \varphi(z) \equiv \ln \xi\left(\frac{1}{1-z}\right) = -\ln 2 + \sum_{n=1}^{\infty}\frac{\lambda_n}{n}z^n = b_0 + \sum_{n=1}^{\infty} b_n(z+1)^n. \tag{1.3}$$

The middle expansion in terms of the Li/Keiper constants λ_n holds for $|z| < \delta_1 < 1$, where $\delta_1 = 1$ corresponds to the Riemann hypothesis. Similarly, the right-most expansion holds for $|z+1| < \delta_2 < 2$. Thus, (1.3) has overlapping domains of expansion, allowing analytic continuation.

In [11], the coefficients b_n of (1.3) are simply an inessential device. However, we treat their properties, including a "curious identity" $b_2 = b_3$, in the next section. The latter relation is simply the beginning of an infinite set of relations that we make explicit. In the Appendix, we record approximate numerical values for the early coefficients. Reference [11] contains physical discussion, including attempting to regard ξ as a quantum mechanical wave function.

To emphasize that the b_n's are not required for the purposes of [11], we have the following argument. Let $N(T)$ be the count of nontrivial zeta zeros for $0 < \text{Im } \rho < T$. We have $N(T) = \pi^{-1}\text{Im} \ln \xi(1/2 + iT)$, and it is well known that as $T \to \infty$, $N(T) = (T/2\pi)\ln(T/2\pi) - T/(2\pi) + O(\ln T)$. Suppose that the RH holds. Then we have

$$\lambda_n = 2\sum_{j=1}^{\infty}(1 - \cos n\theta_j) \geq 0, \tag{1.4}$$

where $\rho = 1/2 \pm i\mu_j$ or $\mu_j = (1/2)\cot(\theta_j/2)$. We can rewrite (1.4) as

$$\lambda_n = 2\int_0^{\infty}[1 - \cos\theta(\mu)]dN(\mu), \tag{1.5}$$

where the lower limit just as well may be taken as μ_1. Integrating by parts, we have

$$\lambda_n = -2n\int_0^{\infty}\sin n\theta \frac{d\theta}{d\mu}N(\mu)d\mu + 2(1 - \cos n\theta)N(\mu)|_0^{\infty}, \tag{1.6}$$

where $\mu(\theta) = \frac{1}{2}\cot(\theta/2)$. For $\mu \to \infty$, we have $\theta = 1/\mu + O(1/\mu^3)$, $1 - \cos n\theta = -n^2/2\mu^2 + O(1/\mu^3)$, and the required limit on the right side of (1.6) is zero. We

therefore obtain the equivalent exact forms

$$\lambda_n = -2n \int_0^\infty \sin n\theta \frac{d\theta}{d\mu} N(\mu) d\mu \tag{1.7}$$

$$= 2n \int_0^{\pi/2} \sin n\theta N(\mu) d\theta(\mu). \tag{1.8}$$

We can note that in (1.8), $\pi/2$ may be replaced by $2\cot^{-1}(2\mu_1) \simeq 1/\sqrt{2}10$, and therefore the range of integration is a relatively narrow one.

We may quickly rewrite (1.7), as $d\theta/d\mu = -4/(4\mu^2+1)$. Furthermore, $\sin n\theta = \sin\theta U_{n-1}(\cos\theta)$, where U_k is the kth Chebyshev polynomial of the second kind, and $\cos\theta = (4\mu^2-1)/(4\mu^2+1)$. We obtain

$$\lambda_n = 32n \int_0^\infty \frac{\mu N(\mu)}{(4\mu^2+1)^2} U_{n-1}\left(\frac{4\mu^2-1}{4\mu^2+1}\right) d\mu, \tag{1.9}$$

thereby recovering (3.13) of [11]. As observed there, on the RH, the values λ_n are indeed nonnegative.

Implicit in [11] (p. 8) are the relations

$$-\ln 2 = b_0 + \sum_{n=1}^\infty b_n, \tag{1.10}$$

and for $n \geq 1$,

$$b_n = \sum_{j=n}^\infty (-1)^{j-n} \binom{j}{n} \frac{\lambda_j}{j}. \tag{1.11}$$

That (1.11) holds may be easily verified by using [11] (3.4) and the orthogonality relation

$$\sum_{m=n}^{j} (-1)^m \binom{m}{n} \binom{j}{m} = (-1)^j \delta_{jn}, \tag{1.12}$$

with δ_{jk} the Kronecker symbol. We shall have recourse to these relations in the following developments.

Relations and representations of b_n

We have

Proposition 1. For $n \geq 1$ we have

$$b_n = \frac{1}{n2^n} \sum_{\rho \in \mathcal{S}} \left[1 - \left(1 + \frac{1}{1-2\rho} \right)^n \right]. \tag{2.1}$$

Corollary 1. We have $b_1 = 0$.

Corollary 2. We have $\zeta'(1/2)/\zeta(1/2) = \frac{1}{2}(\gamma + \frac{\pi}{2} + 3\ln 2 + \ln \pi)$.

In (2.1) the sum includes zeros ρ along with $1-\rho$. (Owing to the functional equation of the ξ function or ζ-functions.) We write $\sum_\rho$ when the companion zero $1-\rho$ is explicitly taken into account.

We have

Corollary 3.

$$b_n = -\frac{1}{n2^n} \sum_{j=1}^{n} \binom{n}{j} \sum_{\rho \in \mathcal{S}} \frac{1}{(1-2\rho)^j}. \tag{2.2}$$

Thus

$$b_n = -\frac{1}{n2^{n-1}} \sum_{k=2}^{[n/2]} \binom{n}{2k} \sum_{\rho} \frac{1}{(1-2\rho)^{2k}}. \tag{2.3}$$

We have

Proposition 2. The coefficient b_{2n+1} is always expressible as a rational linear combination of b_{2n}, b_{2n-2}, ..., b_2.

Examples. We have $b_3 = b_2$, $b_5 = 2b_4 - b_3$, and $b_7 = 3b_3 - 5b_4 + 3b_6$.

Let $\Sigma_{2k} \equiv \sum_{\rho\in\mathcal{S}} \frac{1}{(1-2\rho)^{2k}}$. We have

Corollary 4. We have the relation for n even

$$b_{n+1} = \frac{n}{2} b_n + \frac{1}{(n+1)} \frac{1}{2^n} \sum_{k=1}^{n/2-1} \left[n\binom{n}{2k} - \binom{n}{2k-1} \right] \Sigma_{2k}. \tag{2.4}$$

We have

Proposition 3. We have the summation relation for $n \geq 2$,

$$[1-(-1)^n] b_n = \sum_{j=2}^{n-1} (-1)^j \binom{n-1}{j-1} b_j. \tag{2.5}$$

Corollary 5. In particular, we have $b_2 = b_3$.

Let L_n^α be the Laguerre polynomial of degree n and parameter α [1]. Then we have the following representation.

Proposition 4. We have

$$b_n = \frac{1}{n2^{n+1}} \sum_{\rho\in\mathcal{S}} \int_0^\infty e^{-\rho u} L_{n-1}^1\left(\frac{u}{2}\right) du. \tag{2.6}$$

Corollary 6. On the RH, with $\rho = 1/2 + it_j$, and t_j is real, we have

$$b_n = \frac{1}{n2^n} \sum_{j=1}^\infty \int_0^\infty e^{-u/2} \cos(t_j u) L_{n-1}^1\left(\frac{u}{2}\right) du. \tag{2.7}$$

Write [14]

$$\varphi(z) = 1 + \sum_{j=1}^\infty a_j z^j, \tag{2.8}$$

with $\xi(1/2) = 1 + \sum_{j=1}^\infty (-1)^j a_j = \exp(b_0)$ on the RH. The rapid asymptotic growth of a_j with j has been described in [8]. We have

Proposition 5. We have the recurrence relation for $m \geq 1$

$$\xi(1/2)(m+1)b_m = \sum_{j=m}^\infty (-1)^{j-m}(j+1)\binom{j}{m} a_{j+1} - \sum_{\ell=1}^m \sum_{j=\ell}^\infty (-1)^{j-\ell} \binom{j}{\ell} (m-\ell+1) b_{m-\ell} a_j. \tag{2.9}$$

Proposition 6. On the RH, we have for $n \geq 1$,

$$b_n = 2^{-n} \int_0^{\pi/2} \frac{\sin[(n-1)\theta/2]}{\cos^{n+1}(\theta/2)} N(\mu) d\theta. \tag{2.10}$$

Corollary 7. On the RH, we have for $n \gg 1$,

$$b_n \sim 2^{-n-2} \left[\ln(n-1) + \gamma - 1 - \ln(4\pi)\right]. \tag{2.11}$$

Proposition 7. On the RH, we have

$$\lambda_n = \frac{n}{2} \ln n + (\gamma - 1 - \ln 2\pi)n + o(n). \tag{2.12}$$

A Corollary of Proposition 7 is Corollary 7.

In the next section, proofs are supplied, as well as some discussion.

Proofs of Propositions

Proposition 1. We substitute the sum (1.2) into (1.11),

$$\begin{aligned} b_n &= \sum_{\rho \in \mathcal{S}} \sum_{j=n}^{\infty} \frac{(-1)^{j-n}}{j} \binom{j}{n} \left[1 - \left(1 - \frac{1}{\rho}\right)^j\right] \\ &= \frac{1}{n 2^n} \sum_{\rho \in \mathcal{S}} \left[1 - \left(1 + \frac{1}{1-2\rho}\right)^n\right]. \end{aligned} \tag{3.1}$$

The interchange of sums is justified by the absolute convergence of (1.2).

Corollary 1 immediately follows as we have

$$b_1 = -\frac{1}{2} \sum_{\rho} \left(\frac{1}{1-2\rho} - \frac{1}{1-2\rho}\right) = 0. \tag{3.2}$$

From the Hadamard product for the ξ-function, we have

$$\frac{\xi'(z)}{\xi(z)} = \frac{1}{z} \sum_{\rho \in \mathcal{S}} \frac{1}{1 - \rho/z}. \tag{3.3}$$

Therefore, $b_1 = -\frac{1}{4}\frac{\xi'}{\xi}\left(\frac{1}{2}\right)$, implying Corollary 2.

Corollary 3 (2.2) follows by binomial expansion in (2.1).

Remarks. Corollary 2 recovers what otherwise may be found by applying the functional equation of the ζ function.

Indeed, all odd order derivatives of ξ are zero at 1/2.

Proposition 2. We use (2.3) and put $\Sigma_{2k} \equiv \sum_{\rho\in\mathcal{S}} \frac{1}{(1-2\rho)^{2k}}$. Then, for each n, Σ_{2n} may be eliminated between b_{2n} and b_{2n+1}, and the result follows.

Corollary 4. We have for n even from Corollary 3

$$b_n = -\frac{1}{n2^{n-1}} \sum_{k=1}^{n/2} \binom{n}{2k} \Sigma_{2k}, \tag{3.4a}$$

and

$$b_{n+1} = -\frac{1}{(n+1)2^n} \sum_{k=1}^{n/2} \binom{n+1}{2k} \Sigma_{2k}. \tag{3.4b}$$

Therefore, from (3.4a) we have

$$\Sigma_n = -n2^{n-1}b_n - \sum_{k=1}^{n/2-1} \binom{n}{2k} \Sigma_{2k}. \tag{3.5}$$

We insert this equation into (3.4b) written in the form

$$b_{n+1} = -\frac{1}{(n+1)2^n}\left[\sum_{k=1}^{n/2-1} \binom{n+1}{2k} \Sigma_{2k} + (n+1)\Sigma_n\right]. \tag{3.6}$$

We find

$$b_{n+1} = \frac{n}{2}b_n - \frac{1}{(n+1)}\frac{1}{2^n} \sum_{k=1}^{n/2-1} \left[\binom{(n+1)}{2k} - (n+1)\binom{n}{2k}\right] \Sigma_{2k}. \tag{3.7}$$

Finally, we use a recursion relation for the binomial coefficient, $\binom{m+1}{n} = \binom{m}{n} + \binom{m}{n-1}$, and obtain (2.4).

Examples. We have $\Sigma_4 = 8(-4b_4+3b_3)$, $b_5 = 2b_4-b_3$, $\Sigma_6 = -12(25b_3-40b_4+16b_6)$, and $b_7 = (1/56)(-57b_3 + 80b_4 + 24b_6)$. Of course, $b_3 = b_2 = -(1/4)\Sigma_2$.

Proposition 3. The result is a consequence of the functional equation of the ξ function, so that

$$\xi\left(\frac{1}{1-z}\right) = \xi\left(\frac{z}{z-1}\right) = \varphi(z). \tag{3.8}$$

We have

$$\ln\xi\left(\frac{z}{z-1}\right) = b_0 + \sum_{n=1}^{\infty} b_n \left(1+\frac{1}{z}\right)^n$$

$$= b_0 + \sum_{n=1}^{\infty} b_n (1+z)^n \sum_{k=n-1}^{\infty} (-1)^n \binom{k}{n-1} (1+z)^{k-n+1}, \tag{3.9}$$

using

$$\frac{1}{q^j} = \sum_{k=j-1}^{\infty} \binom{k}{j-1} (1+q)^{k-j+1}. \tag{3.10}$$

Then

$$\ln\xi\left(\frac{z}{z-1}\right) = b_0 + \sum_{n=1}^{\infty} b_n \sum_{k=n}^{\infty} (-1)^n \binom{k-1}{n-1} (1+z)^k$$

$$= b_0 + \sum_{k=1}^{\infty}\sum_{n=1}^{k} b_n (-1)^n \binom{k-1}{n-1} (1+z)^k. \tag{3.11}$$

Comparing with the expansion (1.3), we obtain

$$b_n = \sum_{j=2}^{n} (-1)^j \binom{n-1}{j-1} b_j, \tag{3.12}$$

where we have used $b_1 = 0$. Relation (2.5) follows.

Proposition 4. We use (1.11) and [8] (Prop. 1)

$$\lambda_j = \sum_{\rho\in\mathcal{S}} \int_0^{\infty} e^{-\rho u} L_{j-1}^1(u) du, \tag{3.13}$$

so that

$$b_n = \sum_{\rho \in \mathcal{S}} \int_0^\infty e^{-\rho u} (-1)^n \sum_{j=n}^\infty \frac{(-1)^j}{j} \binom{j}{n} L_{j-1}^1(u) du. \tag{3.14}$$

We use the representation [1] (p. 286) with Bessel function J_α for $\alpha > -1$,

$$L_n^\alpha(x) = \frac{e^x x^{-\alpha/2}}{n!} \int_0^\infty t^{n+\alpha/2} J_\alpha(2\sqrt{xt}) e^{-t} dt, \tag{3.15}$$

giving

$$\sum_{j=n}^\infty \frac{(-1)^j}{j} \binom{j}{n} L_{j-1}^1(u) = e^u u^{-1/2} \sum_{j=n}^\infty \frac{(-1)^j}{j!} \binom{j}{n} \int_0^\infty t^{j-1/2} J_1(2\sqrt{ut}) dt$$

$$= e^u u^{-1/2} \frac{(-1)^n}{n!} \int_0^\infty e^{-2t} t^{n-1/2} J_1(2\sqrt{ut}) dt$$

$$= 2e^u u^{-1/2} \frac{(-1)^n}{n!} \int_0^\infty e^{-2x^2} x^{2n} J_1(2\sqrt{u}x) dx. \tag{3.16}$$

The integral is first evaluated [10] (p. 716) in terms of the confluent hypergeometric function ${}_1F_1$:

$$\sum_{j=n}^\infty \frac{(-1)^j}{j} \binom{j}{n} L_{j-1}^1(u) = \frac{(-1)^n}{2^{n+1}} e^u \, {}_1F_1\left(n+1; 2; -\frac{u}{2}\right)$$

$$= \frac{(-1)^n}{2^{n+1}} \, {}_1F_1\left(1-n; 2; \frac{u}{2}\right)$$

$$= \frac{(-1)^n}{2^{n+1}} \frac{1}{n} L_{n-1}^1\left(\frac{u}{2}\right). \tag{3.17}$$

Here we have used Kummer's first transformation for the function ${}_1F_1$ [1] (p. 191) as well as the relation

$$L_n^\alpha(z) = \binom{n+\alpha}{n} \, {}_1F_1(-n, \alpha+1; z). \tag{3.18}$$

The insertion of (3.17) into (3.14) gives the Proposition.

Remarks. By integrating by parts we may verify that (2.6) returns the sum representation of Proposition 1. We have

$$b_n = \frac{1}{n2^n} \sum_{\rho \in \mathcal{S}} \int_0^\infty e^{-2\rho v} L_{n-1}^1(v) dv$$

$$= -\frac{1}{n2^n} \sum_{\rho \in \mathcal{S}} \int_0^\infty e^{-2\rho v} \frac{d}{dv} L_n(v) dv. \tag{3.19}$$

The integral converges since necessarily Re $\rho > 0$. We use the Laplace transform of a Laguerre polynomial [10] (p. 844), and we recover (2.1).

The theory of Laguerre polynomials is pervasive in formulating the Li criterion [5, 8, 7].

By multiply differentiating (3.17), we have a family of summations,

$$\sum_{j=n}^\infty \frac{(-1)^j}{j} \binom{j}{n} L_{j-k-1}^{k+1}(u) = \frac{(-1)^n}{2^{n+k+1}} \frac{1}{n} L_{n-k-1}^{k+1}\left(\frac{u}{2}\right). \tag{3.20}$$

Otherwise, we may follow the steps as above and find for $\alpha > -1$ and $z \neq 1$

$$\sum_{j=n}^\infty \frac{z^j}{j} \binom{j}{n} L_{j-1}^\alpha(u) = \frac{z^n}{(1-z)^{n+\alpha}} \frac{1}{n} L_{n-1}^\alpha\left(\frac{u}{1-z}\right). \tag{3.21}$$

We have the contour integral representation

$$L_n^\alpha(z) = \frac{\Gamma(n+\alpha+1)}{2\pi i n!} \int_{-\infty}^{(0+)} \left(1 - \frac{z}{t}\right)^n e^t \frac{dt}{t^{\alpha+1}}, \tag{3.22}$$

where the contour encircles the origin in the positive direction and closes at Re $z = -\infty$. This gives

$$\sum_{j=n}^\infty \frac{(-1)^j}{j} \binom{j}{n} L_{j-1}^1(u) = \frac{(-1)^n}{2\pi i} \frac{1}{2^{n+1}} \int_{-\infty}^{(0+)} \frac{(1-u/t)^{n-1}}{(1-u/2t)^{n+1}} e^t \frac{dt}{t^2}. \tag{3.23}$$

It may be verified that the residue at $t = u/2$ gives $L^1_{n-1}(u/2)/n$.

The defining sums of Proposition 1 may be recovered from Corollary 6 in the following way. Write on the RH

$$b_n = -\frac{1}{n2^{n-1}} \sum_{j=1}^{\infty} \int_0^{\infty} e^{-v} \cos(2t_j v) \frac{d}{dv} L_n(v) dv$$

$$= \frac{1}{n2^{n-1}} \sum_{j=1}^{\infty} \left\{ \int_0^{\infty} \left[\frac{d}{dv} e^{-v} \cos(2t_j v) \right] L_n(v) dv + 1 \right\}. \tag{3.24}$$

Then use [10] (p. 846) for n even and odd to evaluate the integrals.

Proposition 5. This follows from the identity $\frac{\varphi'}{\varphi}\varphi = \varphi'$. We make use of

$$1 + \sum_{j=1}^{\infty} a_j z_j = 1 + \sum_{j=1}^{\infty} a_j \sum_{\ell=0}^{j} \binom{j}{\ell} (-1)^{j-\ell} (z+1)^{\ell}$$

$$= 1 + \sum_{j=1}^{\infty} (-1)^j a_j + \sum_{\ell=1}^{\infty} \sum_{j=\ell}^{\infty} (-1)^{j-\ell} \binom{j}{\ell} a_j (z+1)^{\ell}, \tag{3.25}$$

and similarly

$$\varphi'(z) = \sum_{j=1}^{\infty} j a_j z^{j-1} = \sum_{\ell=0}^{\infty} \sum_{j=\ell}^{\infty} (-1)^{j-\ell} (j+1) \binom{j}{\ell} a_{j+1} (z+1)^{\ell}. \tag{3.26}$$

With some further series manipulations we obtain (2.9).

Proposition 6. We have from (1.11) and (1.8)

$$b_n = 2 \sum_{j=n}^{\infty} (-1)^{j-n} \binom{j}{n} \int_0^{\pi/2} \sin j\theta N(\mu) d\theta. \tag{3.27}$$

We then apply the sum $\sum_{j=n}^{\infty} (-1)^j \binom{j}{n} x^j = (-1)^n x^n (1+x)^{-n-1}$ at $x = \exp(\pm i\theta)$. Then simple manipulations yield (2.10).

Corollary 7. In (2.10) we change variable to $x = (n-1)\theta/2$. At the leading order in n we obtain

$$b_n \sim 2^{-n-1} \frac{1}{\pi} \int_0^{\infty} \frac{\sin x}{x} \left[\ln \left(\frac{n-1}{4\pi x} \right) - 1 \right] dx. \tag{3.28}$$

Performing the integral gives (2.11).

Remarks. It is evident that (2.10) includes the cases $b_1 = 0$, $b_2 = b_3$, and $b_5 = 2b_4 - b_3$.

The asymptotic result (2.11) is consistent with the right-most expansion in (1.3) having a radius of convergence at most 2.

Simply from the sum representation (2.1) one may suspect an asymptotic form $b_n \sim \frac{1}{2^n}\frac{1}{\mu_1}$. One could also estimate b_n from

$$b_n = \frac{1}{2\pi i}\int_C \frac{\varphi(z)dz}{(z+1)^{n+1}}, \tag{3.29}$$

where the contour encircles $z = -1$.

For one of the integrals in (3.28), we first have $\int_0^\infty x^\alpha \sin x \, dx = \cos(\pi\alpha/2)\Gamma(\alpha+1)$ for $-2 < \text{Re } \alpha < 0$. Then performing logarithmic differentiation and taking $\alpha \to -1$ we obtain

$$\int_0^\infty \frac{\sin x}{x} \ln\left(\frac{1}{x}\right) dx = \frac{\pi}{2}\gamma. \tag{3.30}$$

Numerically from (2.1) as a sum over the first 10^5 nontrivial zeta zeros [15] we find $b_{1000} \simeq 9.21 \times 10^{-302}$ while from (2.11) we have $b_{1000} \simeq 9.22 \times 10^{-302}$. See figure 1 for a semilog plot of the first 1000 values of b_n versus n (with b_1 omitted). These numerical values, suggesting that indeed the right-most expansion in (1.3) has radius of convergence 2, could be taken as evidence that the RH holds.

Proposition 7. Method 1. Similarly to Corollary 7, we have using (1.8),

$$\lambda_n = 2\int_0^{n/\sqrt{2}10} \sin x \ N\left[\mu\left(\frac{x}{n}\right)\right] dx. \tag{3.31}$$

Therefore, at the leading order we have

$$\lambda_n \sim \frac{n}{\pi}\int_0^\infty \frac{\sin x}{x}\left[\ln\left(\frac{n}{2\pi x}\right)-1\right]dx, \tag{3.32}$$

with the error incurred being $o(n)$. Using (3.30) gives the Proposition.

Method 2. We alternatively use the expression (1.9) and have the expansions

$$\theta(\mu)=\frac{1}{\mu}-\frac{1}{12\mu^3}+O\left(\frac{1}{\mu^5}\right), \tag{3.33}$$

and $\sin(n\theta)=\sin(n/\mu)+O(1/\mu^3)$. We have

$$U_{n-1}(\cos\theta)=\frac{\sin(n\theta)}{\sin\theta}=\frac{(4\mu^2+1)}{4\mu}\sin(n\theta)=\left[\mu+O\left(\frac{1}{\mu}\right)\right]\sin(n\theta). \tag{3.34}$$

We then have

$$\begin{aligned}\lambda_n &\sim 32n\int_{\mu_1}^\infty \frac{\mu^2N(\mu)}{(4\mu^2+1)^2}\sin\left(\frac{n}{\mu}\right)d\mu \sim 2n\int_{\mu_1}^\infty \frac{N(\mu)}{\mu^2}\sin\left(\frac{n}{\mu}\right)d\mu\\ &= 2\int_0^{n/\mu_1} N\left(\frac{n}{v}\right)\sin v dv\\ &\sim -\frac{n}{\pi}\int_0^\infty \frac{\sin v}{v}\left[\ln\left(\frac{2\pi v}{n}\right)+1\right]dv. \end{aligned} \tag{3.35}$$

Using (3.30) we find (2.12).

Corollary 7. We now reprove this Corollary as a result of Proposition 7. We have by (1.11),

$$b_n=\sum_{j=n}^\infty (-1)^{j-n}\binom{j}{n}\left[\frac{1}{2}\ln j+\gamma-1-\ln(2\pi)+o(1)\right]. \tag{3.36}$$

In order to accurately approximate the summand, we use the digamma function $\psi=\Gamma'/\Gamma$, and have $\psi(j)=\ln j+o(1)$ for $j\gg 1$. Then we have, using an integral representation for ψ [10] (p. 943),

$$b_n=\sum_{j=n}^\infty (-1)^{j-n}\binom{j}{n}\left\{\frac{1}{2}[\psi(j)+\gamma]+\frac{\gamma}{2}-1-\ln(2\pi)+o(1)\right\}$$

$$
\begin{aligned}
&= \sum_{j=n}^{\infty}(-1)^{j-n}\binom{j}{n}\left[\frac{1}{2}\int_0^1\left(\frac{t^{j-1}-1}{t-1}\right)dt+\frac{\gamma}{2}-1-\ln(2\pi)+o(1)\right] \\
&= \frac{1}{2}\int_0^1\left[\frac{t^{n-1}}{(t+1)^{n+1}}-\frac{1}{2^{n+1}}\right]\frac{dt}{(t-1)}+\left[\frac{\gamma}{2}-1-\ln(2\pi)\right]\frac{1}{2^{n+1}}+\frac{o(1)}{2^{n+1}} \\
&= \frac{1}{2}\frac{1}{2^{n+1}}\left[\psi(n)+\gamma-\ln 2-\frac{1}{n}\right]+[\frac{\gamma}{2}-1-\ln(2\pi)]\frac{1}{2^{n+1}}+\frac{o(1)}{2^{n+1}} \\
&= \frac{1}{2^{n+2}}\ln n++\left(\gamma-1-\ln\pi-\frac{3}{2}\ln 2\right)\frac{1}{2^{n+1}}+\frac{o(1)}{2^{n+1}}. \qquad (3.37)
\end{aligned}
$$

Remarks. The result (2.12) is not new [5], but we include it and the method of proof as a companion to Proposition 6 and Corollary 7. We suspect that the $o(n)$ terms in (2.12) are of size $O(n^{1/2+\epsilon})$ for any $\epsilon > 0$.

Generally alternating binomial sums may be difficult to estimate, but we have done so in recovering Corollary 7.

Regarding (3.36) and (3.37), it is possible to use an even more accurate approximation to $\ln j$, with $\psi(j+1/2) = \ln j + O(1/j^2)$, but at the cost of a more complicated integral to perform.

Suppose that λ_j has a subdominant term close to $\sqrt{j}$. Then we expect there to be a correction term in b_n close to

$$
\begin{aligned}
&\sum_{j=n}^{\infty}(-1)^{j-n}\binom{j}{n}\frac{1}{j^{1/2}} \sim \sum_{j=n}^{\infty}(-1)^{j-n}\binom{j}{n}\frac{\Gamma(j)}{\Gamma(j+1/2)} \\
&= \frac{\Gamma(n)}{\Gamma(n+1/2)}\ {}_2F_1\left(n,n+1;n+\frac{1}{2};-1\right) \sim \frac{\sqrt{2}}{2^{n+1}}\frac{1}{\sqrt{n}}. \qquad (3.38)
\end{aligned}
$$

Here, ${}_2F_1$ is the Gauss hypergeometric function [1, 10], and by transformation rules [10] (p. 1043), the ${}_2F_1$ function in (3.35) is the same as

$$
2^{-n-1}\ {}_2F_1\left(n,\frac{1}{2};n+\frac{1}{2};\frac{1}{2}\right) = 2^{-n-1}\sqrt{2}\ {}_2F_1\left(-\frac{1}{2},\frac{1}{2};n+\frac{1}{2};-1\right). \qquad (3.39)
$$

The above argument extends so that if λ_j has a subdominant term $j^{1/2+\epsilon}$, we expect in b_n a term close to

$$\sum_{j=n}^{\infty}(-1)^{j-n}\binom{j}{n}\frac{\Gamma(j+\epsilon)}{\Gamma(j+1/2)} = \frac{\Gamma(n+\epsilon)}{\Gamma(n+1/2)}\ {}_2F_1\left(n+\epsilon, n+1; n+\frac{1}{2}; -1\right). \tag{3.40}$$

Appendix: Values of b_n

Exact expressions for b_n can be written in terms of $\ln \pi$, polygammic constants $\psi^{(j)}(1/4)$, and the derivatives $\zeta^{(k)}(1/2)$. The following table gives approximate numerical values for the initial b_n's.

n	b_n
0	-0.698922
1	0
2	0.00144406
3	0.00144406
4	0.00108297
5	0.000721886
6	0.000451088
7	0.00027058
8	0.000157786
9	0.0000901269
10	0.0000506726
11	0.0000281364
12	0.0000154657
13	8.43018×10^{-6}
14	4.56299×10^{-6}
15	2.45502×10^{-6}
16	1.31×10^{-6}
17	6.99×10^{-7}
18	3.71×10^{-7}
19	1.96×10^{-7}
20	1.03×10^{-7}
21	5.44×10^{-8}
22	2.85×10^{-8}
23	1.49×10^{-8}
24	7.79×10^{-9}
25	4.06×10^{-9}

The values $b_0, \ldots, b_{15}$ have been obtained in Mathematica by series expansion of

the φ function, (1.3). The remaining values have been found in Matlab by summing over the first 10^5 complex zeta zeros [15], (2.1).

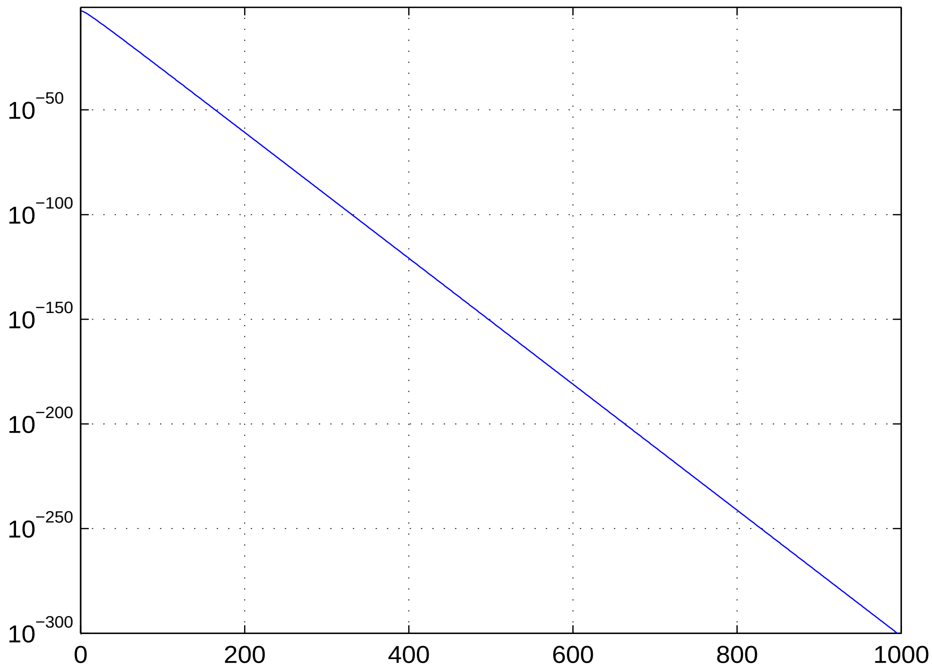

Figure 1: A semilog plot of the first 1000 values of b_n obtained from (2.1) by summing over the first 10^5 complex zeta zeros [15]. The value $b_1 = 0$ is omitted.

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